Intelligent numerical solution of any mathematic problem requires understanding of its mathematical properties ~~con~~ continuous limit.

As with ODE's, numerical solution of PDE's requires an appreciation of their general character and statement of a well-posed problem. Thus, different types of PDE require different solution techniques. We assume knowledge of

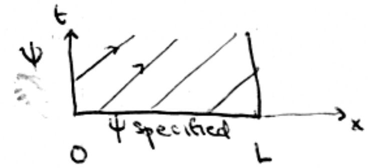
(D1.1-2)

(1) Order of a PDE
(2) Linear vs Nonlinear
(3) Elliptic, Hyperbolic, Parabolic classification of 2^{nd} order ODE's.

In particular, we will often refer to some canonical PDE's:

(a) Advection equation (hyperbolic 1^{st} order)

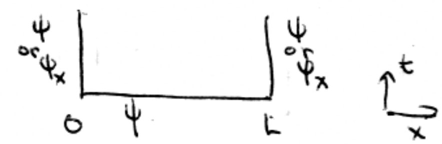
$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0 \quad , \quad 0 \le x \le L$$
$$\psi = f(x-ct) \qquad 0 \le t < \infty$$



(or periodic BC's on ψ)

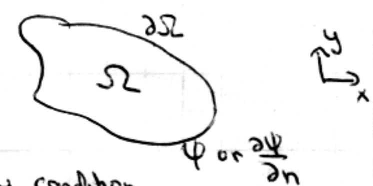
(b) Heat eqn. (parabolic)

$$\frac{\partial \psi}{\partial t} = k\frac{\partial^2 \psi}{\partial x^2} \quad , \quad " \quad "$$



(c) Poisson/Laplace eqn (elliptic)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = F(x,y) \text{ in } \Omega$$
$$\psi \text{ or } \frac{\partial \psi}{\partial n} \text{ specified on } \partial\Omega$$

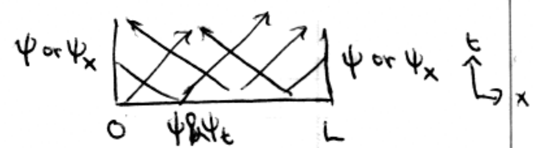


For Neumann BC's $\frac{\partial \psi}{\partial n} = g(x,y)$ on $\partial\Omega$, consistency condition

$$\int_\Omega F\,dV = \int_\Omega \nabla^2\psi\,dV = \int_\Omega \nabla\cdot(\nabla\psi)\,dV = \int_{\partial\Omega} \frac{\partial \psi}{\partial n}\,dS = \int_{\partial\Omega} g\,dS$$

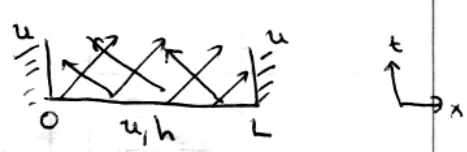
(d) Wave eqn (hyperbolic 2^{nd} order)

$$\frac{\partial^2 \psi}{\partial t^2} - c^2\frac{\partial^2 \psi}{\partial x^2} = 0$$
$$\psi = f(x-ct) + g(x+ct)$$



(e) Shallow water eqns (hyperbolic nonlinear system)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0 \qquad 0 \le x \le L$$
$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \qquad \text{between rigid walls}$$

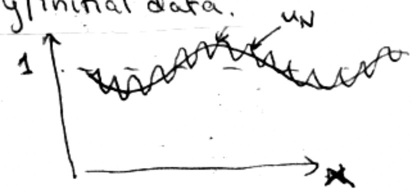


+ Burger's Equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$ (single nonlinear eqn; hyperbolic if $\nu = 0$)

Well-Posedness

A PDE + initial/boundary conditions that represent mathematically a physical problem should be well-posed, i.e.

(1) There should be a solution
(2) It should be unique
(3) It should depend continuously on the boundary/initial data.



Example of ill-posed problem: Backward heat equation

$$u_t = \kappa u_{xx}, \quad -\infty < x < \infty, \quad 0 < t < 1$$
$$u(x,1) = \sin x$$

Find $u(x,0)$.

– There is a unique solution $u(x,t) = e^{\kappa(1-t)} \sin x$. However it does not depend continuously on initial data. To see this, pick an ε. Let $u_N(x,1) = \sin x + \varepsilon \sin Nx$. Then $\max_x |u_N(x,1) - u(x,1)| = \varepsilon$. But no matter how small we make ε, since

$$u_N(x,t) = e^{\kappa(1-t)} \sin x + e^{N^2\kappa(1-t)} \sin Nx,$$

$$u_N(x,0) - u(x,0) = \varepsilon e^{N^2 \kappa} \sin Nx$$

can be made arbitrarily large by taking N large ⇒ not well-posed problem

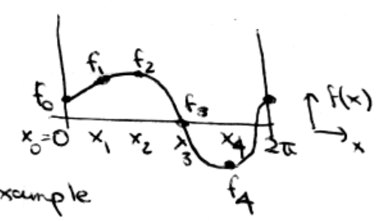
Even a well-posed problem can become numerically ill-posed if numerical method is poorly chosen – we will see examples of this shortly. BCs are a common trap.

Numerical Strategies for PDE's (D1.3)

Must represent continuous function $f(x)$ by a finite set of numbers

Finite Difference Method (FDM)

f described by $f(x_i)$ at a finite set $\{x_i\}$



Derivatives computed using FD approximations to derivatives, e.g. $f'(x_1) = \dfrac{f_2 - f_0}{2\Delta x}$, $\Delta x = 2\pi/5$ in example

Alternatively, f_i may approximate $\displaystyle\int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} f(x)\,dx$ (finite volume method)

Series-expansion Methods

$$f(x, t_n) \approx \sum_{k=1}^{N} c_k(t_n)\, \phi_k(x)$$

is represented by a finite linear combination of fixed expansion functions $\phi_n(x)$.

$$\frac{\partial f}{\partial x}(x,t) \approx \sum_{k=1}^{N} c_k(t_n) \frac{d\phi_k}{dx}$$

For a given $f(x,t_n)$, $c_k(t_n)$ are chosen to minimize some kind of measure of error.

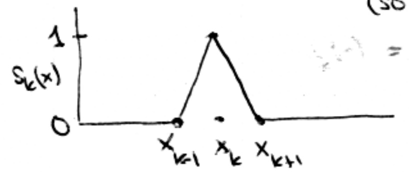
Spectral Method (SM) $\phi_k(x)$ extend over entire domain and are chosen to have nice orthogonality relations & increasing wiggliness

$$f(x) \approx a_1 + a_2 \cos x + a_3 \sin x + a_4 \cos 2x + a_5 \sin 2x$$

Finite Element Method (FEM) $\phi_k(x)$ are localized to nearest grid points (so depend on Δx)

$$f(x) = b_1 s_1(x) + \dots + b_5 s_5(x)$$

$s_k(x) =$ "Chapeau" function



For linear elements, often works similarly to FDM's, but not for higher order elements.

Note that since f is generally not known simultaneously at all times, all time dependent numerical method use FDM for time variations.