AMath 586 / AtmS 581

Multilevel schemes

- Adams-Bashforth
- Leapfrog

Adams-Bashforth schemes

N^{th} order Adams-Bashforth

$$\delta_t^F \phi^n = \frac{\phi^{n+1}-\phi^n}{\Delta t} = \sum_{j=0}^{N-1} c_j F(\phi^{n-j}) \qquad (N^{th} \text{ order accurate})$$

$$\phi' + \frac{\Delta t}{2}\phi'' + \frac{(\Delta t)^2}{3!}\phi''' \dots = \sum_{j=0}^{N-1} c_j \left\{ F - j\Delta t F' + \frac{(j\Delta t)^2}{2!}F'' \dots \right\}$$

Note $\phi' = F$, $\phi'' = F'$, etc., so we use the first N of the constraints

$$1 = \sum_{j=0}^{N-1} c_j$$

$$-\frac{1}{2} = +\sum_{j=0}^{N-1} j c_j$$

$$\frac{1}{3} = \frac{2!}{3!} = \sum_{j=0}^{N-1} j^2 c_j$$

$$\vdots$$

to find $c_0, \dots, c_{N-1}$.

2nd order AB

$$1 = c_0 + c_1$$

$$-\frac{1}{2} = +c_0 + 2c_1 \quad \Rightarrow \quad c_1 = -\frac{1}{2},\ c_0 = \frac{3}{2}$$

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = \frac{3}{2}F(\phi^n) - \frac{1}{2}F(\phi^{n-1})$$

3rd order AB

$$1 = c_0 + c_1 + c_2$$

$$-\frac{1}{2} = c_0 + 2c_1 + 3c_2$$

$$\frac{1}{3} = c_0 + 4c_1 + 9c_2$$

$$\Rightarrow \quad c_0 = \frac{23}{12},\ c_1 = -\frac{16}{12},\ c_2 = \frac{5}{12}.$$

Stability analysis of 2nd order AB, $\quad F(\phi) = \sigma\phi$,

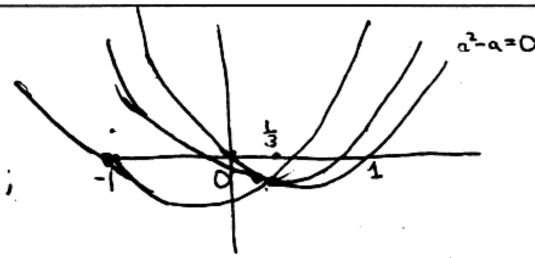
$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = \frac{1}{2}\sigma \{3\phi^n - \phi^{n-1}\}$$

$\phi^n = c_1 a_1^n + c_2 a_2^n$, $a_{1,2}$ are roots of

$$a - 1 = \frac{1}{2}\sigma\Delta t \{3 - a^{-1}\}$$

$$f(a) = a^2 - a\left(1 + \frac{3}{2}\sigma\Delta t\right) + \frac{1}{2}\sigma\Delta t = 0$$

Note that one root has $a_1 \to 1$, the other has $a_2 \to 0$, as $\Delta t \to 0$ (the "physical" and "computational" modes. In general, if σ is real and <0,



$a_1 \downarrow$ as $\sigma \downarrow$, but remains $> \frac{1}{3}$
$a_2 \downarrow$ as $\sigma \downarrow$

When $a_2 < -1$, we have an instability;
this occurs where $f(-1) < 0$,

$$1 + 1 + \tfrac{3}{2}\sigma\Delta t + \tfrac{1}{2}\sigma\Delta t < 0 \quad , \quad \sigma\Delta t < -1$$

For $\sigma = i\kappa$,

$$a^2 - a\left(1 + \tfrac{3}{2} i\kappa\Delta t\right) + \tfrac{i}{2}\kappa\Delta t = 0$$

The 2nd order AB is weakly unstable

$$|a_1| = 1 + \tfrac{1}{4}(\kappa\Delta t)^4$$

and accelerating $R_1 = 1 + \frac{5}{12}(\kappa\Delta t)^2$ on the physical mode.

Normally, method is initialized by a forward difference to get ϕ^1 from ϕ^0; thenceforth the AB scheme is used.

3rd order AB: $|a_1| = 1 - \frac{3}{8}(\kappa\Delta t)^4$

Phase error: $R_1 = 1 + \frac{289}{720}(\kappa\Delta t)^4$ $\Rightarrow$ Weakly damping.

Stable for $\kappa\Delta t < 0.72$

... is very good for oscillation eqns.

Note: AB requires only one function eval. per timestep (an advantage compared to RK)

<u>Leapfrog</u>

$$\delta_{2t}\phi^n = \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} = F(\phi^n) \qquad \text{... 2nd order accurate}$$

Stab. analysis for $F(\phi) = \sigma\phi$ with $\phi^n \propto a^n$,

$$\frac{a^{n+1} - a^{n-1}}{2\Delta t} = \sigma a^n \quad , \quad a^2 - 2\sigma\Delta t\, a - 1 = 0.$$

Again, there is a computational mode.

$$a_1 = \sigma\Delta t + \left((\sigma\Delta t)^2 + 1\right)^{\frac{1}{2}}$$
$$a_2 = \sigma\Delta t - \left((\sigma\Delta t)^2 + 1\right)^{\frac{1}{2}}$$

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Now if $\sigma < 0$, $a_2 < -1$ so the computational mode is always unstable $\Rightarrow$ can't use for decaying solutions. Note for $\sigma = 0$, odd/even decoupl

For oscillations, $\sigma = i\kappa$, $a_1 = e^{i\theta}$, $\theta = \sin^{-1}\kappa\Delta t$, $a_2 = -e^{-i\theta}$.

$\Rightarrow$ $|a_1| = 1$ no amplitude error
$|a_2| = 1$ computational mode not damped

$$R_1 = \frac{\sin^{-1}\kappa\Delta t}{\kappa\Delta t} = 1 + \frac{(\kappa\Delta t)^2}{6} \dots \text{, accelerating.}$$