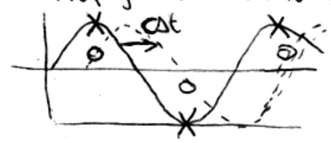
Also note that for $k\Delta x = \pi$, $\phi_{j+\ell} = e^{ikx_{j+\ell}} = (-1)^{j+\ell}$, so the numerical dispersion relation is

2Δx wave: Propagation aliases to dissipation

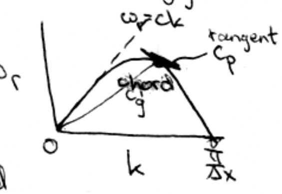


$$-i\omega(-1)^{j} = \sum_{\ell=-\ell_1}^{\ell_2} \alpha_\ell (-1)^{j+\ell}$$

$$\omega = i \cdot C, \text{ where } C = \sum_{\ell=-\ell_1}^{\ell_2} \alpha_\ell (-1)^\ell \qquad (2\Delta x \text{ wave}; k\Delta x = \pi)$$

$$\Rightarrow \quad \mathrm{Re}\,\omega = 0, \quad c_{ph} = \frac{\mathrm{Re}\,\omega}{k} = 0.$$

Thus the $2\Delta x$ wave has zero frequency + phase speed, though it may have substantial dissipation. This leads to unavoidable problems with energy propagation by poorly resolved waves, since the dispersion relation must look at left, so poorly resolved waves not only don't have group velocity $c_g = \frac{\partial \omega}{\partial k} \approx c$, but in fact have negative group velocity and get left behind. This presents problems when simulating spikes and sharp jumps.



<u>Comparison of 4 space-differencing methods on advection eqn</u> (see figs)

1st order (upwind): $D = \delta_x^B \phi_j = \frac{\psi_j - \psi_{j-1}}{\Delta x} = \frac{\partial \psi}{\partial x} - \frac{\Delta x}{2}\psi_{xx} + \ldots$

$$\Rightarrow \quad \omega = \frac{c}{\Delta x}\sin k\Delta x + i(\underbrace{\cos k\Delta x - 1}_{<0 \Rightarrow \text{damping}})$$

2nd order (centered):

$$D = \delta_{2x}\phi_j = \frac{\psi_{j+1} - \psi_{j-1}}{2\Delta x} = \frac{\partial \psi}{\partial x} + \frac{\Delta x^2}{6}\psi_{xxx} + O(\Delta x^3)$$

$$\Rightarrow \quad \omega = \frac{c}{\Delta x}\sin k\Delta x$$

3rd order upwind:

$$D = \frac{2\phi_{j+1} + 3\phi_j - 6\phi_{j-1} + \phi_{j-2}}{6\Delta x} = \frac{\partial \psi}{\partial x} + \frac{\Delta x^3}{12}\psi_{xxxx} + \ldots$$

$$\Rightarrow \quad \omega = \frac{c}{\Delta x}\left[\left(\frac{4}{3}\sin k\Delta x - \frac{1}{6}\sin 2k\Delta x\right) \underbrace{- \frac{i}{3}(1 - \cos k\Delta x)^2}_{<0 \Rightarrow \text{damping}}\right]$$

4th order upwind:

$$D = \frac{4}{3}\delta_{2x}\phi_j - \frac{1}{3}\delta_{4x}\phi_j = \frac{4}{3}\left(\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}\right) - \frac{1}{3}\left(\frac{\phi_{j+2} - \phi_{j-2}}{4\Delta x}\right)$$

$$= \frac{\partial \psi}{\partial x} - \frac{\Delta x^4}{30}\psi_{xxxxx} + O(\Delta x^6)$$

$$\Rightarrow \quad \omega = \frac{c}{\Delta x}\left[\frac{4}{3}\sin k\Delta x - \frac{1}{6}\sin 2k\Delta x\right]$$

Even-order: No dissipation, but strong dispersive errors for $k\Delta x = O(1)$

Odd-order: Dissipative, same dispersive error as next-better even order scheme.

Even better for well-resolved waves, odd for spikes (minimize spurious oscillations).

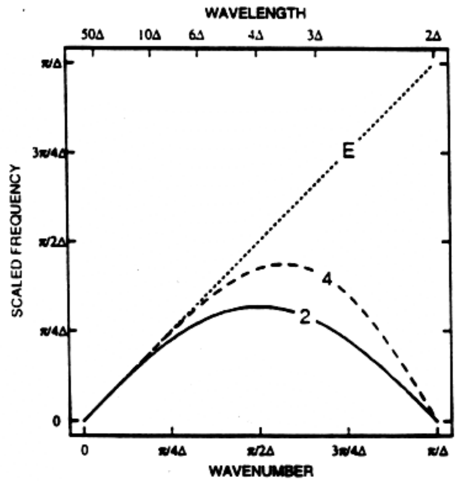


Figure 2.9. Scaled frequency (ω/c) as a function of wavenumber for the analytic solution of the advection equation (dotted line) and for corresponding differential-difference approximations using second- (solid line) and fourth-order (dashed line) centered differences.

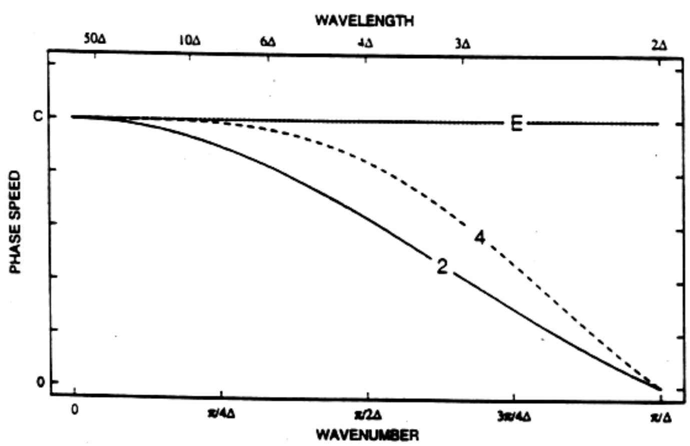


Figure 2.10. Phase speed as a function of numerical resolution for the analytic solution of the advection equation (dotted line) and for corresponding differential-difference approximations using second- (solid line) and fourth-order (dashed line) centered differences.

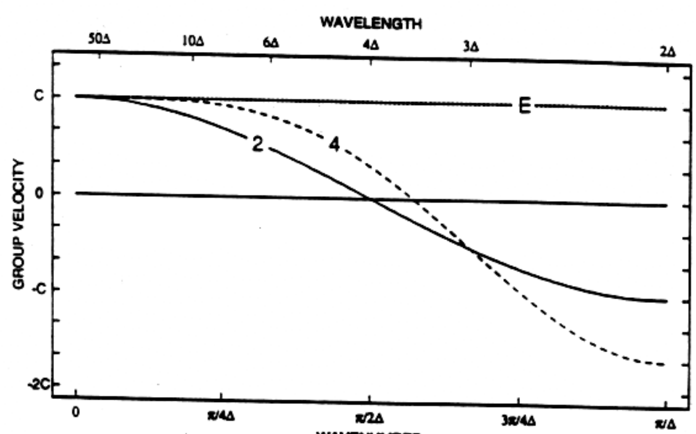


Figure 2.12. Group velocity as a function of numerical resolution for the analytic solution of the advection equation (dotted line) and corresponding differential difference approximations using second- (solid line) and fourth-order (dashed line) centered differences.

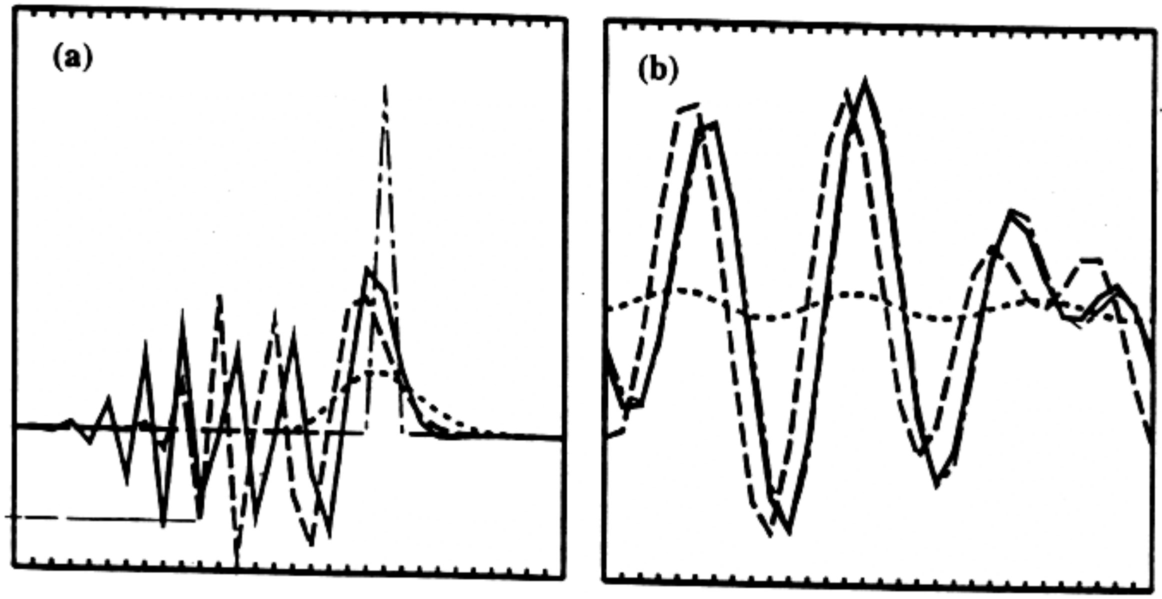


Figure 2.13. Exact solution and differential-difference solutions for (a) advection of a spike over a distance of five grid points, and (b) advection of the sum of equal amplitude 7.5Δx and 10Δx sine waves over a distance of twelve grid points. Exact solution (dot-dashed), one-sided first-order (short-dashed), centered second-order (long-dashed) and centered fourth-order (solid). The distribution is translating to the right. Grid point locations are indicated by the tic marks at the top and bottom of the plot.

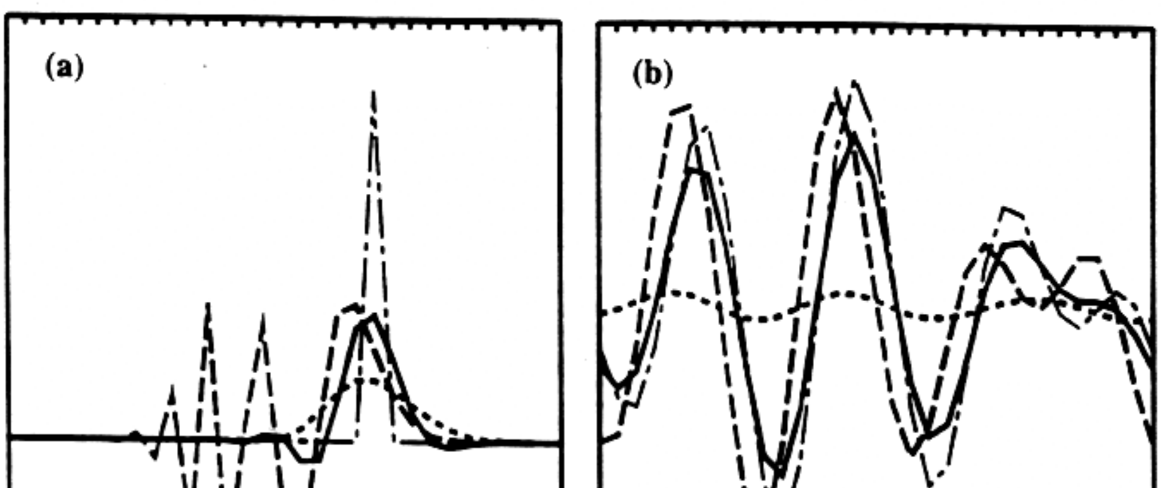


Figure 2.14. Exact solution and differential-difference solutions for (a) advection of a spike over a distance of five grid points, and (b) advection of the sum of equal amplitude 7.5Δx and 10Δx sine waves over a distance of twelve grid points. Exact solution (dot-dashed), one-sided first-order (short-dashed), centered second-order (long-dashed) and one-sided third-order (solid).