Bretherton/586.



## Review of the method of characteristics for 1D quasilinear hyperbolic PDEs:

Consider the 1D advection eqn for unknown $\psi(x,t)$:

$$\psi_t + c(x,t,\psi)\,\psi_x = S(x,t,\psi) \qquad \left(\psi \text{ in } c, S \text{ makes it "quasi" linear}\right)$$

Characteristic form

$$\underbrace{\frac{d\psi}{dt}}_{\psi_t + \frac{dx}{dt}\psi_x} = S(x,t,\psi) \quad \text{on} \quad \underbrace{\frac{dx}{dt} = c(x,t,\psi)}_{\text{characteristic curve}}$$

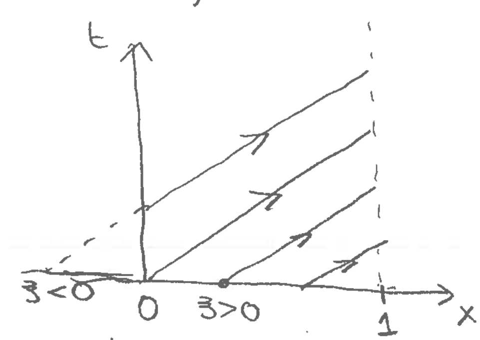
On each characteristic, labeled $\xi$, this is a pair of ODEs that determine $x(t;\xi)$ and $\psi(t;\xi)$ given their values at some point along the characteristic.

Example 1

$$\psi_t + \psi_x = -\psi \qquad 0 < x < 1,\ t > 0$$

$$\psi(x,0) = x$$

$$\psi(0,t) = 0.$$




Characteristic form:

$$\frac{d\psi}{dt} = -\psi \quad \text{on} \quad \frac{dx}{dt} = 1$$

Label each characteristic by its x-intercept $\xi$:

$$C_\xi: \quad \xi = x - t$$

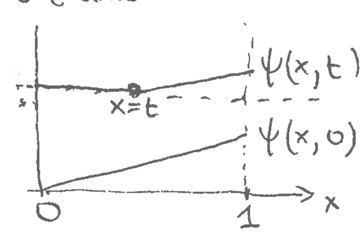
$$\Rightarrow \psi(t;\xi) = \psi_0(\xi)\,e^{-t} \quad \text{on } C_\xi$$

For $\xi > 0$, $C_\xi$ intersects IC $\Rightarrow \psi(0;\xi) = \xi \quad \Rightarrow \psi_0(\xi) = \xi$

For $\xi < 0$, " " " " " BC $\Rightarrow \psi_0(\xi)e^{-t} = 0 \quad \Rightarrow \psi_0(\xi) = 0.$

at time $t = -\xi$ at which $C_\xi$ hits t-axis

$$\Rightarrow \psi(x,t) = \begin{cases} \xi e^{-t}, & \xi = x - t > 0 \\ 0, & \xi = x - t < 0 \end{cases}$$



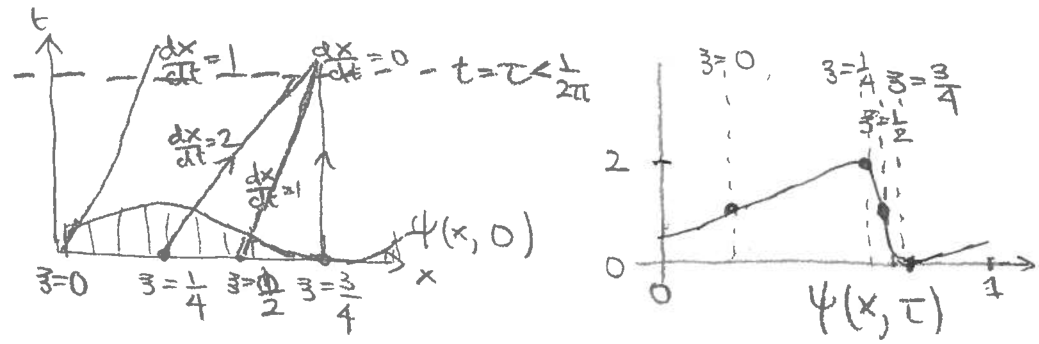
**Example 2** Inviscid Burgers Eqn:

$$\psi_t + \psi\psi_x = 0 \;, \quad 0 < x < 1 \;, \quad \psi(x,0) = 1 + \sin 2\pi x$$

periodic BCs

$$\Rightarrow \frac{d\psi}{dt} = 0 \quad \text{on } C_\xi : \frac{dx}{dt} = \psi$$

$$\Rightarrow \psi(t;\xi) = \text{const. on } x - \psi t = \xi$$
$$= 1 + \sin 2\pi\xi \text{ by IC}$$




formation of a discontinuity or "shock" as characteristics cross.

... one might imagine numerical simulation of shock formation could be challenging.

## Local Stability analysis for PDEs with nonconst coeffs

Consider a PDE with nonconstant coeffs, e.g.

$$\psi_t + \psi\psi_x = 0. \qquad (*)$$

Given a numerical method whose stability properties are known for a constant-coeff analogue to this PDE, how do we analyze its stability on $(*)$?

E.g. upwind method on $\psi_t + c\psi_x = 0$ stable if $0 < \underbrace{\frac{c\Delta t}{\Delta x}}_{\mu} < 1$

How about for $(*)$?

Idea:
- Numerical instabilities develop fastest at shortest resolvable wavelength $2\Delta x$.
- If $\Delta x$ is small, coeffs of PDE are not going to vary much over distances $O(\Delta x)$
- Thus, the instability "sees" the local values of the coefficients and will develop at some location $x_0$ if the local stability threshold based on local values of coeffs at $x_0$ is exceeded.

i.e. for upwind method on $(*)$, stable if $0 < \mu(x_0) < 1$ for all $x_0$, where $\mu(x_0) = \frac{\psi(x_0, t)\Delta t}{\Delta x}$ is local Courant number

so will be stable everywhere if

$$0 < \mu_{max} < 1, \quad \mu_{max} = \max_{x,t} \frac{\psi(x,t)\Delta t}{\Delta x}.$$