Finite Differencing (D2.1 ; assumed review)

$$f(x_0+\Delta x) = f(x_0) + \Delta x\, f_x(x_0) + \frac{(\Delta x)^2}{2} f_{xx}(x_0) \ldots$$

$$f_x(x_0) = \underbrace{\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}_{\text{forward difference}} + \underbrace{\frac{(\Delta x)}{2} f_{xx}(x_0) \ldots}_{\text{local truncation error (LTE)} = O(\Delta x)}$$

If truncation error is $O(\Delta x^m)$, method is m'th order accurate.

Difference operators make for compact notation.

$$\delta_{nx} f(x) = \frac{1}{n\Delta x}\left\{ f\left(x + \frac{n\Delta x}{2}\right) - f\left(x - \frac{n\Delta x}{2}\right) \right\}$$ centered difference of span $n\Delta x$

$$\delta_{nx}^F f(x) = \frac{1}{n\Delta x}\left\{ f(x+n\Delta x) - f(x) \right\}$$

$$\delta_{nx}^B f(x) = \frac{1}{n\Delta x}\left\{ f(x) - f(x-n\Delta x) \right\}$$

$$\delta_{2x} f = f_x(x) + \frac{(\Delta x)^2}{6} f_{xxx}(x) + O(\Delta x)^4$$ 2nd order approx to f_x

$$\delta_x^2 f = \delta_x\left\{ \frac{f(x+\frac{\Delta x}{2}) - f(x-\frac{\Delta x}{2})}{\Delta x} \right\} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

$$= f_{xx} + \frac{(\Delta x)^2}{12} f_{xxxx} + O(\Delta x)^4 \ldots$$

They also allow higher-order accurate formulas to be derived, e.g.

$$f_x(x) = \delta_{2x} f - \frac{(\Delta x)^2}{6} f_{xxx}(x) + O((\Delta x)^4)$$

$$= \delta_{2x} f - \frac{(\Delta x)^2}{6}\left(\delta_x^2 f_x + O(\Delta x^2)\right) + O(\Delta x^4)$$

$$= \left\{ 1 - \frac{(\Delta x)^2}{6}\delta_x^2 \right\} \delta_{2x} f + O(\Delta x^4)$$

<u>Boundary conditions</u>

Heat Eqn: $\frac{\partial\psi}{\partial t} = k\frac{\partial^2\psi}{\partial x^2}$ $\quad 0 \le x \le L$

$$\psi(x,0) = f(x)$$

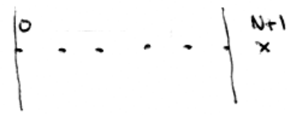
$$\cancel{\partial}\psi(0,t) = g(t)$$

$$\frac{\partial\psi}{\partial x}(L,t) = 0.$$

Discretize in x only for now: at $x_k = k\Delta x$, $k=0,\dots,N$, $\Delta x = \frac{L}{N}$:

$$\frac{\partial\psi_k}{\partial t} = k\,\delta_x^2\psi_k + O((\Delta x)^2),\ k=1,\dots,N-1$$

$$\psi_0(t) = g(t)$$



For $\psi_N(t)$, two alternatives:

(1) Use a <u>one-sided</u> formula, e.g.

$$\left(\frac{\partial^2\psi}{\partial x^2}\right)_N = \delta_x^B\left(\frac{\partial\psi}{\partial x}\right)_N + O(\Delta x)$$

$$= \frac{\cancelto{0}{\left(\frac{\partial\psi}{\partial x}\right)_N} - \left(\frac{\partial\psi}{\partial x}\right)_{N-1}}{\Delta x} + O(\Delta x)$$

$$\approx -\frac{\delta_x\psi_{N-1}}{\Delta x} + O(\Delta x) \quad \text{... only 1st order accurate.}$$

(1b) Use a 2nd order accurate one-sided formula

$$\left(\frac{\partial^2\psi}{\partial x^2}\right)_N = \left[\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)\right]_N$$

$$\delta_x^B f = \frac{f(x) - f(x-\Delta x)}{\Delta x} = f'(x) - \frac{\Delta x}{2}f''(x) + O((\Delta x)^2)$$

$$\delta_{2x}^B f = \frac{f(x) - f(x-2\Delta x)}{2\Delta x} = f'(x) - \Delta x f''(x) + O((\Delta x)^2)$$

$$\Rightarrow \quad f'(x) = 2\delta_x^B f - \delta_{2x}^B f + O(\Delta x)^2$$

$$\Rightarrow \quad \left(\frac{\partial^2\psi}{\partial x^2}\right)_N = 2\frac{\left(\frac{\partial\psi}{\partial x}\right)_{N-1} - \cancelto{0}{\left(\frac{\partial\psi}{\partial x}\right)_N}}{\Delta x} - \frac{\left(\frac{\partial\psi}{\partial x}\right)_{N-2} - \cancelto{0}{\left(\frac{\partial\psi}{\partial x}\right)_N}}{2\Delta x} + O(\Delta x)^2$$

Now find third order accurate formulas for $\left(\frac{\partial\psi}{\partial x}\right)_{N-1}$ and $\left(\frac{\partial\psi}{\partial x}\right)_{N-2}$ ... complicated.

(2) Use an <u>image point</u> x_{N+1}

B.C. $\Rightarrow$ $\delta_{2x}\psi_N = 0$ to $O(\Delta x^2)$ $\Rightarrow$ $\psi_{N+1} = \psi_{N-1} + O(\Delta x^4)$

and now all points are defined.

$$\left(\frac{\partial^2\psi}{\partial x^2}\right)_N = \frac{\psi_{N+1} - 2\psi_N + \psi_{N-1}}{(\Delta x)^2} = \frac{2(\psi_N - \psi_{N-1})}{(\Delta x)^2} = -\frac{\delta_x^F\psi_{N-1}}{\Delta x} + O(\Delta x^2).$$

Periodic BC's: $\psi_0 = \psi_N$, $\psi_{N+1} = \psi_1$