Fourier spectral simulation of 2D fluid flow

Problem statement

Consider 2D incompressible inviscid fluid flow in a periodic domain $0 < x, y < 1$. The flow is described by a streamfunction $\psi(x, y, t)$ such that $(u, v) = (-\psi_y, \psi_x)$. If we define the vertical vorticity

$$\zeta(x, y, t) = v_x - u_y \ = \psi_{xx} + \psi_{yy}, \tag{1}$$

the flow evolves according to the vorticity equation

$$D\zeta/Dt = \zeta_t - \psi_y\zeta_x + \psi_x\zeta_y = 0. \tag{2}$$

The initial streamfunction is an elongated isolated vortex

$$\psi(x, y, 0) = -0.25\exp([-4(x - 0.5)^2 - (y - 0.5)^2]/2\sigma^2)$$

where $\sigma = 0.15$.

A Fourier pseudospectral method is ideal for accurately solving this problem because the domain is periodic and the expected solution is fairly smooth so the derivatives can be accurately calculated spectrally.

Given ζ and ψ at time t, the basic method is to step ζ forward to time $t + \Delta t$ using (2). Then we regard (1) as a Poisson equation with periodic BCs for updating ψ to time $t + \Delta t$.

Fourier spectral solution

We implement a pseudospectral method with RK4 time differencing to solve the vorticity equation. As for the 1D advection equation, a CFL stability restriction can be derived from the RK4 stability region of RK4 applied to the amplification equation $dq/dt = \sigma q$, where σ is a complex growth rate. For a wavelike disturbance $\exp(iK[x\cos\theta + y\sin\theta - Ut]$ of speed U and angular propagation direction θ spectrally advected though the domain, $\sigma = -iKU$. For imaginary sigma, the RK4 stability region goes out to —Im($\sigma\Delta t$)—< 2.82. The maximum wavenumber supportable on the grid has $\max(k_x) = \pi/\Delta x$ and similarly for $\max(k_y)$, giving a maximum diagonal wavenumber of $K_{max} = \max(k_x^2 + k_y^2)^{1/2} = 2^{1/2}\pi/\Delta x$. Thus in 2D the CFL stability condition is

$$2.82 > K_{max}U_{max}\Delta t = 2^{1/2}\pi U_{max}\Delta t/\Delta x$$

from which we deduce $U_{max}\Delta t/\Delta x < 0.63$, where U_{max} is the maximum fluid velocity anywhere in the domain. This is a bit stricter than the 1D stability condition $U_{max}\Delta t/\Delta x < 0.9$.

Matlab script `FS_vortex.m` implements a 2D pseudospectral method for the 2D inviscid vorticity eqn. We use a 2D DFT in x and y (`fft2` in Matlab) with $N = 64$ Fourier modes in each direction, and a timestep $\Delta t = 0.25\Delta x$, where the grid spacing $\Delta x = 1/N$. This timestep was chosen by trial and error. It briefly exceeds the CFL limit for the velocities of up to 2.9 that occur early in this simulation; instability doesn't occur because these are not exactly diagonal to the grid, but we are living dangerously with this large a timestep. Other timestep choices would be optimal for other simulations.

The script produces the sequence of plots in Fig. 1. In the plots, negative contours are dashed and positive contours are solid. The contour interval on all plots is 0.05 for ψ and 10 for ζ. One can see the initial vortex being rotated counterclockwise and deformed by the flow, stretching part of the vorticity into streamers. The vorticity starts to develop a ragged structure at the grid scale at later times and becomes underresolved at this N around $t = 0.5$. The streamfunction looks much smoother since it is an inverse Laplacian of the vorticity, and largely just rotates counterclockwise with minor distortions in shape.

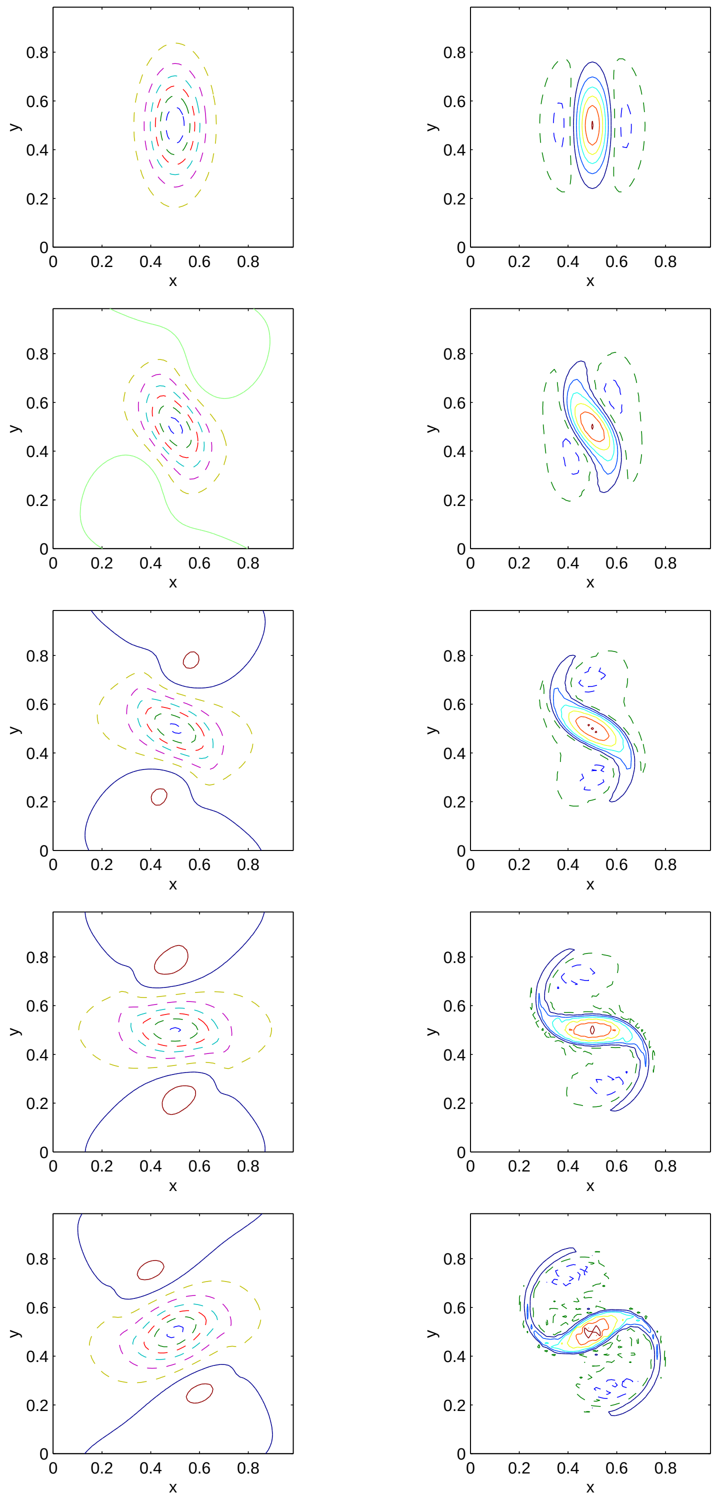

Figure 1: Streamfunction(left) and vorticity(right) at t = 0, 0.125, 0.25, 0.375, 0.5. Max velocity U_{max} = 2.88 at $t = 0.125$

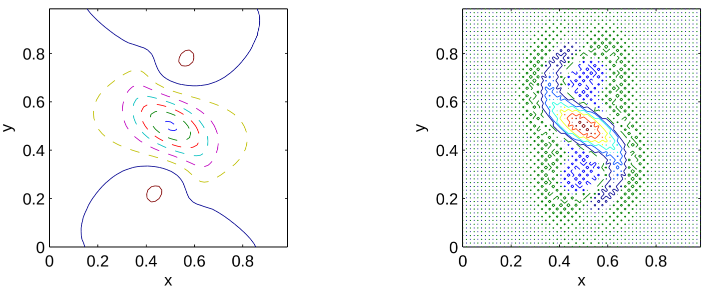

Figure 2: Solution with $D_4 = 3$ at t = 0.125, showing numerical instability due to overrelaxation of high wavenumbers by excessive 4th-order damping.

Use of 4th order smoothing to damp grid-scale noise in ζ

The ragged structure of ζ occurs because vorticity is being fluxed to high wavenumbers by the effect of nonlinear advection. It can be smoothed in a physically realistic way by damping the highest wavenumbers. In a real fluid, this occurs through molecular diffusion $\nu\nabla^2\zeta$. However, in a numerical simulation of 'inviscid' flow, a 4th-order smoother $-\nu_4\nabla^4\zeta$ more concentrated on the highest wavenumbers is often considered preferable, as this leaves the motions at larger scales more nearly unaltered while still serving as a drain. This, like 2nd-order diffusion, is trivial to implement spectrally, since $\nabla^4 \leftarrow (k_x^2 + k_y^2)^2$ in the Fourier domain. It may be viewed as a very simple parameterization of subgrid 2D turbulence on the vorticity field. The damping does not change the relation of vorticity to streamfunction, nor does it affect the periodic BCs.

We choose ν_4 to damp the highest resolvable wavenumbers by a large fraction each timestep, so that nonlinear flux of vorticity into these wavenumbers is strongly damped. For the highest wavenumber K_{max}, the damping rate associated with the 4th-order smoother is $\sigma_{max} = -\nu_4 K_{max}^4$. To keep the RK4 timestepping stable, we must restrict $-\sigma_{max}\Delta t < 2.79$ (the constant for RK4 stability of a purely decaying model is nearly the same as for oscillations). This mandates that

$$\nu_4 < 2.79/(K_{max}^4\Delta t)$$

In our Matlab script, we include an option for 4th order damping specified through a parameter D_4 such that $\nu_4 = D_4/(K_{max}^4\Delta t)$. We must choose $D_4 < 2.79$ for stability. Fig. 2 shows that if $D_4 = 3$ is chosen larger than this, instability in ζ rapidly develops. On the other hand, Fig. 3 shows that if $D_4 = 2$, the solution at $t = 0.5$ is free of fine-scale vorticity noise as desired.

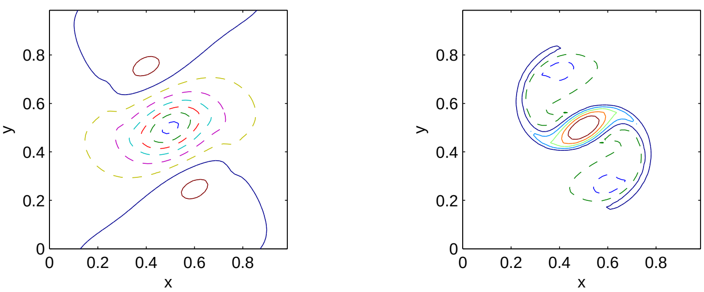

Figure 3: Solution with $D_4 = 2$ at t = 0.5, showing desirable smoothing of numerical solution for ζ by smaller 4th-order damping.

Sensitivity to $\Delta t/\Delta x$

Next, we show the timestep sensitivity. Fig. 4 shows the solution at $t = 0.5$ for our standard setup $N = 64$ and $D_4 = 2$, but now with twice the timestep such that $\Delta t/\Delta x = 0.5$. The domain-maximum simulated fluid velocity U_{max} at this time is given at the top of the left panel. The solution is still stable even though $U_{max}\Delta t/\Delta x = 0.875$ exceeds the 2D CFL stability limit, because the high velocity region moves around the domain before the highest wavenumbers have a chance to blow up. However, with $\Delta t/\Delta x = 0.55$ the solution blows up around $t = 0.15$.

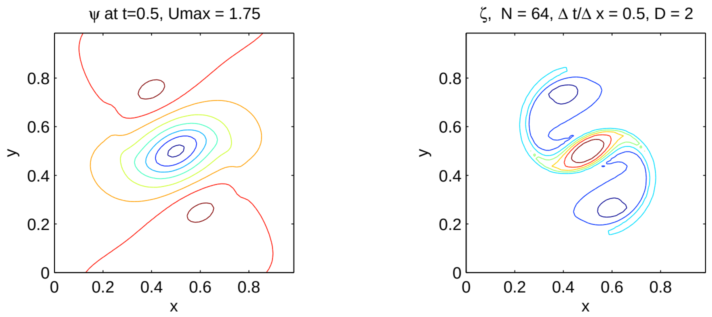

Figure 4: Solution with $N = 64, D_4 = 2$ is stable out to $t = 0.5$ even for twice as large a timestep with $\Delta t/\Delta x = 0.5$.

Sensitivity to number of modes N

Fig. 5 shows the sensitivity of the $D_4 = 2$ simulation to the grid resolution or equivalently the number of Fourier modes. The $N = 16$ solution is quite inaccurate, but for $N = 32$ and greater, only small changes are visible and only in the vorticity field, which is more concentrated at high wavenumbers.

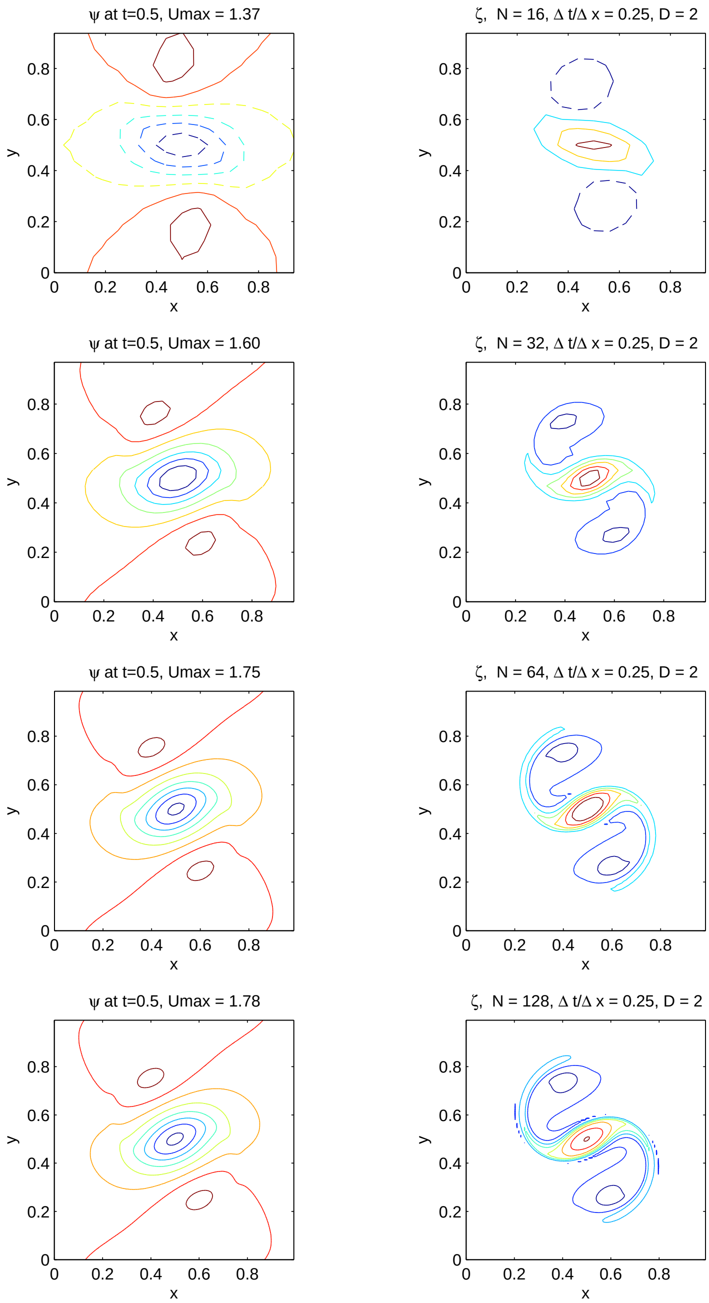

Figure 5: Resolution convergence of solution at t = 0.5.