Combined space/time differencing; consistency

$L[\psi] = \frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$, $c > 0$. Let FDA to $\psi(x_j, t_n)$ be ϕ_j^n

Upstream scheme: $\delta_t^F \phi_j^n + c\delta_x^B \phi_j^n = L_a[\phi_j^n] = 0$ $\qquad \frac{\partial}{\partial t} \leftarrow \delta_t^F$, $\frac{\partial}{\partial x} \leftarrow \delta_x^B$

If exact solution is substituted into FDM,

$$L_a[\psi^n]_j^n = \delta_t^F \psi_j^n + c\delta_x^B \psi_j^n = \cancel{\psi_t} + \frac{\Delta t}{2}\psi_{tt} + \dots + c\left(\cancel{\psi_x} + \frac{\Delta x}{2}\psi_{xx} + \dots\right)$$

$$= \underbrace{\frac{\Delta t}{2}\psi_{tt} + c\frac{\Delta x}{2}\psi_{xx} \dots}_{T_a[\psi]\text{, local truncation error.}}$$

If truncation error is $O(\Delta x^p, \Delta t^q)$ method is p'th order accurate in t and q'th order in x. If $T_a[\psi] \to 0$ as $\Delta x, \Delta t \to 0$ the scheme is called **consistent**.

(D2.2) A FDM is **convergent** to p^{th} order in t, q^{th} order in x if

$$\|\psi(x_j, t) - \phi_j^n\| = O(\Delta t^p) + O(\Delta x^q) \qquad \text{as } \Delta t, \Delta x \to 0$$

for all $t_i < T$, where T is some positive time independent of $\Delta x, \Delta t$, where $\|\phi\|_p = \left(\sum_{j=1}^N |\phi_j|^p \Delta x\right)^{1/p}$

($p = 2 \to$ euclidean norm, $p = \infty \to$ max norm)

(Recall properties of norms).

A FDM is **stable** if

$$\|\phi^n\| \le C_T \|\phi^0\| \quad \text{for } n\Delta t \le T\text{, a fixed positive time, and all sufficiently small } \Delta x, \Delta t.$$

Exponentially stable if $C_T \le Ae^{BT}$, strictly stable if $C_T \le 1$.

Lax Equivalence Thm (Lax & Richtmeyer 1956)

If a FDM is linear, stable and accurate of $O(\Delta t^p, \Delta x^q)$ it is convergent of $O(\Delta t^p, \Delta x^q)$

Testing for stability of a FD scheme

(i) Energy method

The PDE being approximated often has some form of 'energy' conservation. A well-designed numerical scheme will often preserve this conservation law, which may ensure stability.

E.g.

$$\psi_t + c\psi_x = 0, c>0 \quad \psi(0)=\psi(1)$$

$$\Delta x = \frac{1}{N}$$

FDA: $\psi(x_j, t_n) \approx \phi_j^n$

$$\delta_t^F \phi_j^n + c\delta_x^B \phi_j^n = 0 \quad , \phi_0^n = 0, \ \phi_N^n = 0.$$

$$\Rightarrow \phi_j^{n+1} = \mu\phi_{j-1}^n + (1-\mu)\phi_j^n, \ \mu = \frac{c\Delta t}{\Delta x} = \text{Courant number}$$

Cons. law for PDE:

$$\frac{\partial}{\partial t}\int_0^1 \psi^2 dx = \int_0^1 2\psi\psi_t dx = -2c\int_0^1 \psi\psi_x dx = 0.$$

$\Rightarrow$ Try FDA analogue:

$$\frac{\sum_{j=1}^{N}(\phi_j^{n+1})^2\Delta x - \sum_{j=1}^{N}(\phi_j^n)^2\Delta x}{\Delta t} = \frac{\Delta x}{\Delta t}\sum_{j=1}^{n}\left[\left\{\phi_j^n - \mu(\phi_j^n - \phi_{j-1}^n)\right\}^2 - \phi_j^{n\,2}\right]$$

$$= \frac{\Delta x}{\Delta t}\sum_{j=1}^{n}\left[-2\mu\underbrace{\phi_j^n}_{\frac{1}{2}(\phi_j+\phi_{j-1})+\frac{1}{2}(\phi_j-\phi_{j-1})}(\phi_j^n - \phi_{j-1}^n) + \mu^2(\phi_j^n - \phi_{j-1}^n)^2\right]$$

$$= \frac{\Delta x}{\Delta t}\sum_{j=1}^{n}\left[-\mu(\phi_j^{n\,2} - \phi_{j-1}^{n\,2}) - \mu(\phi_j - \phi_{j-1})^2 + \mu^2(\phi_j^n - \phi_{j-1}^n)^2\right]$$

$$\frac{\Delta x}{\Delta t}\left\{\|\phi^{n+1}\|_2^2 - \|\phi^n\|_2^2\right\} = -\frac{\Delta x}{\Delta t}\cdot\mu(1-\mu)\sum_{j=1}^{n}(\phi_j^n - \phi_{j-1}^n)^2 \le 0 \text{ if } 0 \le \mu \le 1$$

This approach is limited but can handle BC's (and show how to handle BC's to maintain stability)