Von Neumann Method (D2.2.2)

- Linear FDA with constant coefficients, periodic BC's $\phi(0)=\phi(L)$, or unbounded.
- Based on writing $\phi_j^n = \sum_{m=-M}^{M} a_m^n e^{ikx_m}$ $\quad x_m = m\Delta x$

or unbounded.

- Note that in this case, if $\phi_j^{n+1} = P\phi_j^n$, where P is the propagator operator for one timestep, then in general e^{ikx} will be an eigenfunction of P:

$$P\{e^{ikx_j}\} = A(k)e^{ikx_j}$$

where $A(k)$ is called the amplification factor. Thus if $\phi_j^0 = e^{ikx_j}$, $\phi_j^n = \underbrace{A^n(k)}_{B_n(k)}\phi_j^0$. Clearly a necessary condition for exponential stability is L5

that $|A^n(k)| \le e^{\beta n\Delta t}$ as $\Delta t \to 0$, for any $k = \frac{2\pi m}{L}$, $m=0,1,\ldots$ resolvable on grid.

$|A(k)| \le 1 + \beta\Delta t$, β ind. of $\Delta t, \Delta x$. } Von Neumann stability conditions

For strict stability, $|A(k)| \le 1$.

Von Neumann Analysis of $\psi_t + c\psi_x = 0$.

$$\delta_t^F \phi_j^n + c\,\delta_x^B \phi_j^n = 0 \quad , \quad c>0$$

If $\phi_j^n = e^{ikx_j}$, $\phi_j^{n+1} = A(k)e^{ikx_j}$, then

$$\left\{ \frac{A(k)-1}{\Delta t} + c\left(\frac{1-e^{-ik\Delta x}}{\Delta x}\right)\right\} = 0.$$

$$A(k) = 1 - \mu\left(1-e^{-ik\Delta x}\right) = 1 - ik\mu\Delta x - \frac{(\Delta x)^2}{2}\mu k^2 \ldots$$

Note that if $A(k)$ were exact, it would be $A_{ex}(k) = e^{-ikc\Delta t}$

$$= 1 - ikc\Delta t - \frac{k^2c^2(\Delta t)^2}{2} + O((\Delta t)^3)$$

$$A_{ex}(k) = 1 - ik\mu\Delta x - \frac{k^2\mu^2(\Delta x)^2}{2} + O(\mu^3(\Delta x)^3).$$

Thus $A(k) = A_{ex}(k) + O((\Delta x)^2)$ as we expect for a 1st-order accurate method.

Now $\quad |A(k)|^2 = \left[1-\mu(1-e^{-ik\Delta x})\right]\left[1-\mu(1-e^{ik\Delta x})\right] = 1-2\mu(1-\mu)(1-\cos k\Delta x)$

Thus $|A(k)| \le 1$ if $\quad 2\mu(1-\mu) \geq 0$ or $0 \le \mu \le 1$, as before. The wave that amplifies fastest when $\mu > 1$ or $\mu < 0$ has $\cos k\Delta x = -1$, $m = \frac{N}{2}$, the "$2\Delta x$" wave.

Note that if $\mu > 1$, the $2\Delta x$ wave has $A(k) = 1 - 2\mu(1-\mu)\cdot 2 = (2\mu - 1)^2$

Thus in n timesteps (at a time $n\Delta t$), where $n = T/\Delta t$, if $\phi_j^n = e^{ikx_j}$, $k = \frac{\pi}{\Delta x}$,

$$\|\phi^n\| / \|\phi^0\| = |A(k)|^n = (2\mu-1)^{2n} \rightarrow \infty \text{ as } \Delta t \to 0 \quad \Rightarrow \text{method unstable.}$$