Amplitude/phase errors (D2.3.1)

If the exact amplification factor during a time Δx for $\psi = e^{ikx}$ is $A_e(k)$

(i.e. if $L[\psi] = 0$, $-\infty < x < \infty$,

$\psi(x,0) = e^{ikx}$, then $\psi(x,\Delta t) = A_e(k) e^{ikx}$).

then if

$$a(k) = \frac{A(k)}{A_e(k)},$$ is the relative amplification rate

$|a(k)| - 1$ is the amplitude error

$$\left(\begin{array}{lll} |a(k)| < 1 & \Rightarrow & \text{"damping"} \\ > 1 & \Rightarrow & \text{"amplifying"} \\ = 1 & \Rightarrow & \text{"neutral"} \end{array} \right)$$

and if $R = \frac{\arg(A)}{\arg(A_e)}$ is the relative phase change

$$\left(\begin{array}{lll} R < 1 & \Rightarrow & \text{"decelerating"} \\ > 1 & \Rightarrow & \text{"accelerating"} \end{array} \right.$$

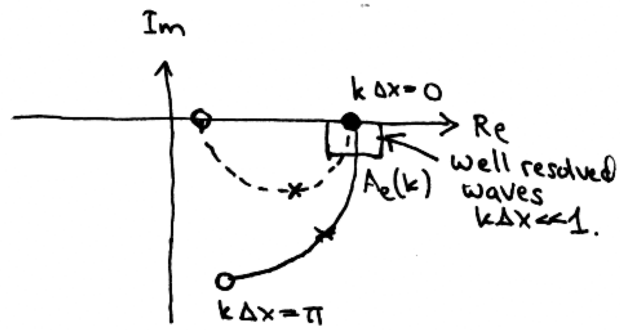
For upwind method on advection eqn:

$$A(k) = 1 - \mu(1 - e^{-ik\Delta x}) \quad , \quad A_e(k) = \exp\{-ikc\Delta t\} = \exp\{-ik\mu\Delta x\}$$

$$= 1 - \mu(1 - \cos k\Delta x) - i\mu \sin k\Delta x$$

We can plot both $A_e(k)$ and $A(k)$ for wavenumbers $0 < k < \frac{\pi}{\Delta x}$ resolvable on the grid. Consider the case $\mu = 0.4$ for definiteness.

Use •, x, o to show A_e, A at $k\Delta x = 0, \frac{\pi}{2}, \pi$ in the complex plane:



$A_e(k)$ traces out a circle of radius 1 out to argument $-\mu\pi$ at $k\Delta x = \pi$.

$A(k)$ traces out a circle of radius μ and center $1-\mu$, going from 1 to $1-2\mu$ in a clockwise directions as $k\Delta x$ ~~goes~~ from 0 to π.

"Well-resolved" wavenumbers with $\lambda = \frac{2\pi}{k} \gg \Delta x \Rightarrow k\Delta x \ll 1$ are in the rectangle at right. For this regime, A_e and A can be Taylor-expanded in $k\Delta x$ for more information.

Graphically, $\left|\frac{A(k)}{A_e(k)}\right| \le 1$ for all k $\Rightarrow$ upwind method is <u>damping</u> (so stable) for $\mu = 0.4$

Note that for $\mu < 0$ or $\mu > 1$ this would not be the case.

For $k\Delta x \ll 1$, can show

$$\left|\frac{A}{A_e}\right| = 1 - \mu(1-\mu)(k\Delta x)^2 + O([k\Delta x]^4).$$

Graphically (look at $k\Delta x = \frac{\pi}{2}$), $\frac{\arg(A)}{\arg(A_e)} < 1 \Rightarrow$ method is <u>decelerating</u>

Note particularly that for the $2\Delta x$ wave ($k\Delta x = \pi$), $\arg A = 0$ (no wave propagation), compared to $\arg(A_e) = -\mu\pi$, a particularly large phase error.

For $k\Delta x \ll 1$, can show

$$\frac{\arg A}{\arg A_e} = 1 + \left(-\frac{\mu^2}{3} + \frac{\mu}{2} - \frac{1}{6}\right)(k\Delta x)^2 + \ldots$$

$\Rightarrow$ decelerating for $\mu < \frac{1}{2}$

accelerating for $\mu > \frac{1}{2}$.

Discrete dispersion relation (D2.5)

If we expect wavelike or exponential behavior from the soln (as we do for constant coefficient PDE's) we can cast $A(k)$ in terms of a discrete dispersion relation

$$\phi_j^n = e^{i[kx_j - \omega t_n]}$$

$$\Rightarrow \quad A(k) = e^{-i\omega\Delta t}, \quad \omega = \log\left(\frac{A(k)}{-i\Delta t}\right) = \omega(k, \Delta x, \Delta t)$$

For the upwind method, the exact dispersion for the advection eqn is

$$\omega_e = ck$$

and the discrete dispersion is

$$e^{-i\omega\Delta t} = 1 - \mu\left(1 - e^{-ik\Delta x}\right) \qquad (*)$$

For advection equation if $\omega = \omega_r + i\omega_i$ is the soln of $(*)$

$$\operatorname{Im}\omega_i \begin{cases} < 0 & \text{damping} \\ = 0 & \text{neutral} \\ > 0 & \text{~~unstable~~ amplifying} \end{cases}$$

$$\frac{\omega_r}{\omega_e} \begin{cases} > 1 & \text{accelerating} \\ < 1 & \text{decelerating} \end{cases}$$

This concept can be particularly useful for understanding well resolved ($k\Delta x \ll 1$) solutions, since in this limit the dispersion relation simplifies, e.g. for upwind

$$e^{-i\omega\Delta t} = \left\{1 - \mu\left(1 - \left[1 - ik\Delta x - \frac{(k\Delta x)^2}{2} \ldots\right]\right)\right\}$$

$$= 1 - i\mu k\Delta x + \frac{\mu(k\Delta x)^2}{2} \ldots \qquad \approx 1$$

$$\Rightarrow \quad -i\omega\Delta t = \log\left\{1 + \left(-i\mu k\Delta x - \mu\frac{(k\Delta x)^2}{2} \ldots\right)\right\}$$

$$= \left(-i\mu k\Delta x - \mu\frac{(k\Delta x)^2}{2} \ldots\right) - \frac{(\quad)^2}{2} \ldots$$

$$= -i\mu k\Delta x + (k\Delta x)^2\left\{\frac{\mu^2 - \mu}{2}\right\} \ldots$$

Since $\Delta t = \frac{\mu}{c}\Delta x$,

$$\omega = \frac{\mu k\Delta x}{\frac{\mu}{c}\Delta x} - \frac{i(k\Delta x)^2}{\frac{\mu}{c}\Delta x} \cdot \frac{\mu(1-\mu)}{2}$$

$$\omega = ck - \frac{ic\Delta x}{2} \cdot (1-\mu)k^2, \quad k\Delta x \ll 1 \quad \text{(damping, no rel. phase error to } O(\Delta x))$$

$-i\omega \Leftrightarrow \frac{\partial}{\partial t}$, $ik \Leftrightarrow \frac{\partial}{\partial x}$

$$\Longleftrightarrow \quad \frac{\partial\psi}{\partial t} = -c\frac{\partial\psi}{\partial x} + \frac{c\Delta x}{2}(1-\mu)\frac{\partial^2\psi}{\partial x^2} \qquad \text{(Modified equation)}$$